One Class of Dual Matrix Methods

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ABSTRACT

One class of dual matrix methods is given, dependent on three parameters which can be chosen arbitrarily. It is proven that the minimum of a quadratic form with n variables will be obtained in at most $n+1$ steps in the case of an arbitrary choice of parameters. This method class enables concrete methods to be developed. Finally, it is shown how two known dual matrix methods are obtained as special cases.

1. INTRODUCTION

Methods not using linear minimization and enabling the minimum of a quadratic form with a positive definite symmetric Hesse matrix to be determined in finite steps have only begun to be dealt with in recent years. This is accounted for by the fact that for nonquadratic functions the linear minimizations are very operation intensive. This led to the development of the method of dual matrices, in which field two papers have appeared so far [3, 4].

For reasons similar to those of Broyden [1] and Huang [5] in the case of quasi-Newton methods, we intend to construct a class of dual matrix methods. We note, in addition, that the method class considered also contains processes of nonsymmetric type, but in this connection we refer to the paper of Broyden, Dennis and Moré [2], in which the authors show that in the "direct prediction" methods the nonsymmetric Pearson method is locally convergent, but at the same time there exists a rank-1 symmetric method which is not. As we do not use linear minimization in the method class considered, we think that the nonsymmetric procedures can also be of interest.

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2. DESCRIPTION OF ONE CLASS OF DUAL MATRIX METHODS

In order to understand the method of dual matrices, let us start from the paper of Huang [4].

Let us consider the following quadratic function:

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c \qquad (\mathbf{x},\mathbf{b}\in R^n, \quad c\in R^1), \tag{2.1}$$

where A is a positive semidefinite symmetric $n\times n$ matrix with $r(A)=m\leqslant n$. The gradient of $f[\mathbf{x}]$ at the point $\mathbf{x}$ is

$$\mathbf{g}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}. \tag{2.2}$$

Let $\mathbf{y}_i=\mathbf{g}_{i+1}-\mathbf{g}_i$ and $\mathbf{s}_i=\mathbf{x}_{i+1}-\mathbf{x}_i$, where $\mathbf{g}_i$ is the gradient of $f(\mathbf{x})$ at the point $\mathbf{x}_i\in R^n$; the series $\mathbf{x}_i$, $i=1,2,\dots,n$, will be defined later.

It follows from (2.2) that

$$\mathbf{y}_i = A\mathbf{s}_i \qquad (i=1,2,\dots,n). \tag{2.3}$$

As the rank of A is $m\leqslant n$, there obviously exist at most m linearly independent $\mathbf{y}_i$.

Let the series $\mathbf{x}_i$, $i=1,2,\dots,n$, be determined so that for the directions $\mathbf{s}_i$, $i=1,2,\dots,n$,

$$\mathbf{s}_i^T A\mathbf{s}_j = 0 \qquad (1\leqslant i\leqslant j-1) \tag{2.4}$$

should be satisfied, i.e., the direction $\mathbf{s}_i$ should be A-conjugated.

Assume that $\mathbf{g}_l$ obtained in the lth step is a linear combination of the previous $\mathbf{y}_i$ directions. Then we can get in the next step the minimum of $f(\mathbf{x})$. As

$$\mathbf{g}_l = \sum_{i=1}^{l-1} c_i\mathbf{y}_i \qquad \left(c_i\in R^1, i=1,2,\dots,l-1\right), \tag{2.5}$$

we have, because of the equality (2.3),

$$\mathbf{g}_i = \sum_{i=1}^{l-1} c_i A\mathbf{s}_i.$$

By multiplicating this from left by s_i^T and using (2.4), we obtain that

$$g_l = \sum_{i=1}^{l-1} \frac{s_i^T g_l}{s_i^T y_i} A s_i.$$

Let

$$s_l = -\sum_{i=1}^{l-1} \frac{s_i s_i^T}{s_i^T y_i} g_l. \tag{2.6}$$

Then for g_{l+1} we obtain the following:

$$g_{l+1} = g_l + A s_l = \sum_{i=1}^{l-1} \frac{s_i^T g_l}{s_i^T y_i} A s_i - A \sum_{i=1}^{l-1} \frac{s_i s_i^T}{s_i^T y_i} g_l = 0.$$

Then in fact $x_{l+1} = x_l + s_l$ is the minimum point of $f(x)$, because

$$g_{l+1} \equiv 0.$$

It follows from the above that if $m = n$, then we get the minimum after at most $n+1$ steps. It is clear from the relation (2.6) that in order to determine s_l we need a matrix series defined the following way:

$$B_{i+1} = B_i + \frac{s_i s_i^T}{s_i^T y_i} \qquad (i = 1, 2, \ldots, n) \tag{2.7}$$

with B_1 the zero matrix.

It can be seen at once that this matrix series has the following properties:

$$B_{i+1} y_j = s_j \qquad (1 \leqslant j \leqslant i, \quad i = 1, 2, \ldots, n) \tag{2.8}$$

On the basis of the property (2.8) we define a matrix series more general than the previous one.

Let us seek now B_{i+1} in the form $B_{i+1} = B_i + C_i$. Hence

$$B_{i+1} y_j = B_i y_j + C_i y_j = s_j \qquad (j = 1, 2, \ldots, i). \tag{2.9}$$

Note the following property of the matrix C_i. Denote by Y_j $(1 \leqslant j < i-1)$ the matrix obtained from the vectors y_j. Then $C_i Y_j = 0$ $(j = 1, 2, \ldots, i-1)$, because of the conditions (2.8) and (2.9). Let the directions z_j, g_j be such that

$$z_j^T y_j = 1, \quad q_j^T y_j = 1 \qquad (j=1,2,\ldots,i), \tag{2.10}$$

and define the matrix C_i as follows:

$$C_i = s_j q_j^T - B_i y_j z_j^T.$$

The above mentioned property of matrix C_i holds, of course, in case of such a choice, too.

We require directions z_j, q_j to be such that (2.10) is fulfilled. Let, e.g.,

$$q_j^T = \frac{1-\beta_j y_j^T B_j y_j}{s_j^T y_j} s_j^T + \beta_j y_j^T B_j \qquad (\beta_j \in R^1, \quad j=1,2,\ldots) \tag{2.11}$$

and

$$z_j^T = \frac{\delta_j y_j^T B_j y_j + 1}{s_j^T y_j} s_j^T - \delta_j y_j^T B_j \qquad (\delta_j \in R^1, \quad j=1,2,\ldots). \tag{2.12}$$

By using (2.11) and (2.12) we obtained the following:

$$B_{j+1} = B_j + \frac{1-\beta_j y_j^T B_j y_j}{s_j^T y_j} s_j s_j^T + \beta_j s_j y_j^T B_j - \frac{\delta_j y_j^T B_j y_j + 1}{s_j^T y_j} B_j y_j s_j^T + \delta_j B_j y_j y_j^T B_j. \tag{2.13}$$

THEOREM 2.1. *The choice of the matrix sequence B_j according to* (2.13) *with arbitrary constants β_j, δ_j satisfies the condition* (2.8).

Proof. It follows at once from the relation (2.10), the expression (2.9) and the definition of C_i for an arbitrary index $j \geqslant 1$ that

$$B_{j+1} y_j = s_j. \tag{2.14}$$

Assume now that

$$B_j y_i = s_i \qquad (1 \leqslant i < j) \tag{2.15}$$

and note that (2.15) remains correct if $j \to j+1$. By making use of the definition, the above described property of C_i, the choice of the vectors $\mathbf{q}_j, \mathbf{z}_j$ according to the condition (2.10), and the property (2.14), we obtain that

$$\mathbf{B}_{j+1}\mathbf{y}_i = B_j\mathbf{y}_i + C_j\mathbf{y}_i = B_j\mathbf{y}_i = \mathbf{s}_i, \qquad (1 \leqslant i \leqslant j) \tag{2.16}$$

by which we have realized the statement of Theorem 2.1. ■

Note that the choice of the weaker limitations

$$\mathbf{z}_j^T y_j \neq 0 \quad \text{and} \quad \mathbf{q}_j^T \mathbf{y}_j \neq 0,$$

instead of the conditions (2.10), leads us back to the conditions (2.10) after all, for because of the property of the matrices C_i following from (2.9) and the definition of C_i, the two constants have to coincide.

Furthermore the question arises whether in the expressions (2.11) and (2.12) $\mathbf{q}_j^T$ or $\mathbf{z}_j^T$ could be chosen as arbitrary linear combinations of the vectors $\mathbf{s}_j$ and $\mathbf{y}_j^T B_j$. Let e.g.

$$\mathbf{q}_j^T = \varepsilon_j \mathbf{s}_j^T + \vartheta_j \mathbf{y}_j^T B_j.$$

Because of the condition corresponding to (2.10), the equality

$$\mathbf{q}_j^T \mathbf{y}_j = \varepsilon_j \mathbf{s}_j^T \mathbf{y}_j + \vartheta_j \mathbf{y}_j^T B_j \mathbf{y}_j = 1$$

has to be fulfilled for every j; therefore the relation

$$\varepsilon_j = \frac{1 - \vartheta_j \mathbf{y}_j^T B_j \mathbf{y}_j}{s_j^T \mathbf{y}_j}$$

must hold between ε_j and ϑ_j, which requires the choice (2.11).

The set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ has to be determined so that the directions $\mathbf{s}_1, \mathbf{s}_2, \ldots$ determined from them satisfy the condition (2.4). We seek the set of points $\{\mathbf{x}_i\}$ in the following form.

Let H_1 be a positive definite matrix, $\mathbf{x}_1 \in R^n$ an arbitrary vector, and

$$\left.\begin{aligned} &\mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{s}_i, \\ &\mathbf{s}_i = -\alpha_i H_i^T g_i, \quad \alpha_i \neq 0, \ \alpha_i \in R^i \end{aligned}\right\} \qquad (i = 1, 2, \ldots).$$

We do not assume for α_i that is is minimizing in the direction $\mathbf{s}_i$.

The condition (2.4) can be transformed in the following way:

$$s_i^T A s_j = -\alpha_i g_i^T H_i y_j = 0. \tag{2.17}$$

Thus the matrix H_i has to be constructed so that

$$H_i y_j = \mathbf{0} \qquad (1 \leqslant j \leqslant i-1, \quad i=2,3\ldots) \tag{2.18}$$

holds; then to any direction $g_i \neq \mathbf{0}$ all $H_i y_j$ will be orthogonal. The matrix H_i too is sought in the recursive form $H_i = H_{i-1} + D_i$.

Because of the condition (2.18),

$$H_{i-1} y_j + D_{i-1} y_j = \mathbf{0} \qquad (0 \leqslant j \leqslant i-2, \quad i=2,3,\ldots)$$

Let the direction w_j be such that the equality $w_j^T y_j = 1$ holds for every j, and define D_{i-1} in the following way:

$$D_{i-1} = -H_{i-1} y_j w_j^T.$$

Let w_j^T be as follows:

$$w_j^T = -\gamma_j s_j^T + \frac{\gamma_j s_j^T y_j + 1}{y_j^T H_j y_j} y_j^T H_j. \tag{2.19}$$

Then

$$H_i = H_{i-1} + \gamma_{i-1} H_{i-1} y_{i-1} s_{i-1}^T - \frac{\gamma_{i-1} s_{i-1}^T y_{i-1} + 1}{y_{i-1}^T H_{i-1} y_{i-1}} H_{i-1} y_{i-1} y_{i-1}^T H_{i-1}. \tag{2.20}$$

THEOREM 2.2. *If H_1 is a positive definite matrix, then the matrix set determined by* (2.20) *satisfies the condition* (2.18).

Proof. First we notice that

$$H_2 y_1 = \mathbf{0}. \tag{2.21}$$

According to the definition,

$$H_2 y_1 = H_1 y_1 + \gamma_1 H_1 y_1 s_1^T y_1 - \frac{\gamma_1 s_1^T y_1 + 1}{y_1^T H_1 y_1} H_1 y_1 y_1^T H_1 y_1 = \mathbf{0}. \tag{2.22}$$

Assume that

$$H_i \mathbf{y}_k = \mathbf{0} \qquad (1 \leqslant k \leqslant i-1) \tag{2.23}$$

exists, and note that (2.23) holds also when $i \to i+1$. This takes place in two parts. By applying (2.22) to substitute the index 2 by $i+1$ and the index 1 by i, it follows that

$$H_{i+1} \mathbf{y}_i = \mathbf{0}. \tag{2.24}$$

It is sufficient therefore to realize that

$$H_{i+1} \mathbf{y}_k = \mathbf{0} \qquad (1 \leqslant k < i).$$

By utilizing (2.20) we obtain that

$$H_{i+1}\mathbf{y}_k = H_i\mathbf{y}_k + \gamma_i H_i \mathbf{y}_i \mathbf{s}_i^T \mathbf{y}_k - \frac{\gamma_i \mathbf{s}_i^T \mathbf{y}_i + 1}{\mathbf{y}_i^T H_i \mathbf{y}_i} H_i \mathbf{y}_i \mathbf{y}_i^T H_i \mathbf{y}_k \qquad (1 \leqslant k < i).$$

Because of the inductive assumption (2.23) $H_i \mathbf{y}_k = \mathbf{0}$, on the other hand,

$$\mathbf{s}_i^T \mathbf{y}_k = -\alpha_i \mathbf{g}_i^T H_i \mathbf{y}_k = 0$$

also holds, and therefore

$$H_{i+1} \mathbf{y}_k = \mathbf{0} \qquad (1 \leqslant k < i) \tag{2.25}$$

The equalities (2.24) and (2.25) prove the statement of the theorem. ■

On this basis we can construct a general class of the dual matrices which determines the minimum of the quadratic function $f(\mathbf{x})$ in at most $m+1$ steps, and this class does not require any linear minimization along the direction—only a nonzero step.

Let H_1 be a positive definite matrix, $\mathbf{x}_1 \in R^n$ be such that $\mathbf{g}(\mathbf{x}_1) \neq \mathbf{0}$ and B_1 be an arbitrary matrix. Then the class is defined by the following algorithm:

$$\mathbf{s}_i = -\alpha_i H_i^T \mathbf{g}_i. \tag{2.26}$$

Here $\alpha_i \neq 0$, and for an arbitrary nonquadratic function such that $f(\mathbf{x}_{i+1}) < f(\mathbf{x}_i)$ we require

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \mathbf{s}_i, \tag{2.27}$$

$$H_{i+1} = H_i + \gamma_i H_i \mathbf{y}_i \mathbf{s}_i^T - \frac{\gamma_i \mathbf{s}_i^T \mathbf{y}_i + 1}{\mathbf{y}_i^T H_i \mathbf{y}_i} H_i \mathbf{y}_i \mathbf{y}_i^T H_i, \tag{2.28}$$

$$B_{i+1} = B_i + \frac{1 - \beta_i \mathbf{y}_i^T B_i \mathbf{y}_i}{\mathbf{s}_i^T \mathbf{y}_i} \mathbf{s}_i \mathbf{s}_i^T + \beta_i \mathbf{s}_i \mathbf{y}_i^T B_i$$

$$- \frac{\delta_i \mathbf{y}_i^T B_i \mathbf{y}_i + 1}{\mathbf{s}_i^T y_i} B_i \mathbf{y}_i \mathbf{s}_i^T + \delta_i B_i y_i y_i^T B_i, \tag{2.29}$$

and if the direction $\mathbf{g}_l$ is a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{l-1}$, then

$$\mathbf{s}_l = -B_l \mathbf{g}_l. \tag{2.30}$$

As the algorithm has free parameters, it is possible to choose the parameters for each step.

3. SYMMETRIC CASE OF THE GENERAL SCHEME

In this section that case of the general scheme will be described when both matrix sets H_i and B_i are symmetric.

If $\gamma_i = 0$ $(i = 1, 2, \ldots)$, then instead of (2.28) we obtain

$$H_{i+1} = H_i - \frac{H_i \mathbf{y}_i \mathbf{y}_i^T H_i}{\mathbf{y}_i^T H_i \mathbf{y}_i}. \tag{3.1}$$

If, on the other hand, $\beta_i = -(\delta_i \mathbf{y}_i^T B_i \mathbf{y}_i + 1)/\mathbf{s}_i^T \mathbf{y}_i$, then

$$B_{i+1} = B_i + \frac{1 - \beta_i \mathbf{y}_i^T B_i \mathbf{y}_i}{\mathbf{s}_i^T \mathbf{y}_i} \mathbf{s}_i \mathbf{s}_i^T + \beta_i \left(\mathbf{s}_i \mathbf{y}_i^T B_i + B_i \mathbf{y}_i \mathbf{s}_i^T \right) - \frac{\beta_i \mathbf{s}_i^T \mathbf{y}_i + 1}{y_i^T B_i y_i} B_i y_i y_i^T B_i. \tag{3.2}$$

Thus in the symmetric case the matrix set H_i is determined unambiguously, while in the matrix set B_i the parameter β_i can be chosen arbitrarily.

4. DERIVATION OF THE KNOWN METHODS FROM THE GENERAL SCHEME

The dual matrix methods described in the foregoing are symmetric. Every method uses the expression (3.1) to determine the matrix set H_i. If in (3.2) $\beta_i = 0$ $(i = 1, 2, \ldots)$, then we get Algorithm II in Huang's paper [4], in which

$$B_{i+1} = B_i + \frac{s_i s_i^T}{s_i^T y_i} - \frac{B_i y_i y_i^T B_i}{y_i^T B_i y_i}.$$

If $\beta_i = 1/(s_i - B_i y_i)^T y_i$, then we get Algorithm III in Huang's paper, in which

$$B_{i+1} = B_i + \frac{(s_i - B_i y_i)(s_i - B_i y_i)^T}{(s_i - B_i y_i)^T y_i}.$$

The subsequent known algorithms present themselves in a way quite similar to the foregoing.

5. CONDITION FOR THE CHOICE OF STEP (2.30) AND CONTINUATION IN THE CASE OF NONQUADRATIC FUNCTIONS

As the linear dependence of the gradient direction on the directions $y_1, y_2, \ldots$ can be decided on the computer only within a certain accuracy, we have to specify a tolerance $\varepsilon > 0$ in advance. In his paper Huang obtained the following condition for this.

If for an index l

$$\left| \frac{g_l^T s_l}{g_l^T g_l} \right| < \varepsilon,$$

then we have to continue the algorithm with the step (2.30). As in the case of a nonquadratic function, the procedure probably does not come to an end with this step; we have to determine the continuing matrices H_l, B_l too. As the matrices H_1, B_1 are positive definite, therefore, if the matrix set B_l is symmetric and $\beta_i \leqslant 0$, we can make the choice

$$H_{l+1} = B_l,$$

$$B_{l+1} = B_l.$$

The expression (3.2) determines just the Broyden class, whose positive definiteness was proved by Broyden. As the Broyden class is obtained by the choice $\beta_i = -\vartheta_i$, we get a positive definite and symmetric matrix set if

$$\beta_i \leqslant 0$$

In every other case we can make the choice

$$H_l = B_1,$$

$$B_l = B_1.$$

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